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Refocusing of a Time-Reversed Acoustic Pulse Propagating in Randomly Layered Media

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In this paper, we study the propagation of acoustic waves in a layered random medium. Using the time-reversed method, we prove the space-time refocusing effect and the stabilization for acoustic signals. This work is a generalization of ref. 4 and 7 in the three-dimensional case.

KEY WORDS: Acoustic waves; random media; reflected velocity; time reversal method; stochastic equations.

1. INTRODUCTION

A time-reversal mirror is, roughly speaking, a device which is capable of receiving an acoustic signal in time, keeping it in memory, and sending it, or a part of it, back into the medium in the reversed direction of time. Time-reversal mirrors have been developed in the context of ultrasounds by Mathias Fink and his team at the Laboratoire Ondes et Acoustique (ESPCI-Paris). They have studied experimentally their effects and proposed various applications. One of the striking effects is the refocusing property which can be described as follows: an acoustic pulse, initially like a point source, which propagates through a random medium produces a transmitted signal with a coda. If a part of this coda is time-reversed and sent back into the same medium one observes a refocusing at the place of the initial source. Moreover it seems that randomness improves this refocusing effect. In recent works, Clouet–Fouque[(4)] and Ndzié,[(7)] have analysed mathematically this effect in a one-dimensional model and in the context of separation of scales between the correlation length of the inhomogeneities

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present in the medium, the typical wavelengths of the pulse and the distance of propagation as introduced in ref. 1. In the one-dimensional case only the refocusing effect in time is present. This work is a generalization of ref. 4 to the three-dimensional layered case. In the framework of separation of scales for randomly layered media,[1] we derive mathematically by applying simultaneously the approximation-diffusion theorem[6] and the stationary phase theorem,[2] the space-time refocusing effect and the stabilization for time-reversed acoustic signals. Furthermore, we have introduced a variation of scales in the definition of the time-reversed pulse. This technique provides important information on the relation between the position of the time-reversal mirror, the portion of the signal used as the new pulse on the one hand, and the shape of the new reflected signal on the other hand.

The structure of this paper is as follows. In the next section, the model of the problem as well as the definitions of some functions are given. In Section 3, we give the equations of amplitudes with their boundary values. Our main result on refocusing properties is stated and proved in Section 4. Section 5 contains the stabilization of the time-reversal wave, and in the appendix, we give a version of the approximation-diffusion theorem.

2. FORMULATION OF THE PROBLEM

2.1. The Acoustic Equations in Layered Media

We consider acoustic wave propagation in a three-dimensional space described by the linearized equations of momentum and mass conservation for the velocity $\vec{u}(t, x, y, z)$ and pressure $p(t, x, y, z)$,

$$\begin{cases} \rho(z)\,\dfrac{\partial \vec{u}}{\partial t} + \nabla p = F(t, \mathbf{x})\, \delta_{(z-z_s)}\vec{e} \\[2ex] \dfrac{1}{K}\dfrac{\partial p}{\partial t}(z) + \nabla\cdot\vec{u} = 0 \end{cases} \tag{1}$$

where $\rho = \rho(z)$ and $K = K(z)$ are respectively the density and bulk modulus of the medium layered in the vertical z direction and $\mathbf{x} = (x, y)$ is the horizontal offset. We assume a vertical source, $\vec{e} = (0, 0, 1)$, at depth $z_s < 0$ but near the surface $\{z = 0\}$, with a smooth compactly supported pulse-shape function $F(t, \mathbf{x})$. We also assume that the medium is randomly varying in a slab $\{-L < z < 0\}$ of thickness L and homogeneous above the surface $\{z > 0\}$ and below the slab $\{z < -L\}$. Only the bulk modulus K is varying inside of the slab as described in the next section, the density ρ being constant:

$$\frac{1}{K(z)} = \begin{cases} \dfrac{1}{K_0} & \text{if} \quad z > 0 \\ \dfrac{1}{K_2} & \text{if} \quad z < -L \end{cases} \qquad \text{and} \qquad \rho(z) = \begin{cases} \rho_0 & \text{if} \quad z > 0 \\ \rho_1 & \text{if} \quad -L < z < 0 \\ \rho_2 & \text{if} \quad z < -L \end{cases} \tag{2}$$

In the derivation of the boundary conditions for these equations (Section 3.3) we shall assume that the source is inside of the slab near the interface $\{z = 0\}$ and that the density ρ_0 above this interface is much smaller than the density ρ_1 just below.

2.2. Separation of Scales

We will adopt the separation of scales introduced by G. Papanicolaou and his co-workers in ref. 1. More precisely, we will assume that the wavelengths are of order ε, the correlation length of order ε^2 and macroscopic variations in order one; with ε a small non-negative real. It is in these scales that the pulse acts as a probe.

We define the bulk modulus in the slab $\{-L < z < 0\}$ as:

$$\frac{1}{K(z)} = \frac{1}{K_1(z)} \left(1 + \eta \left(z, \frac{z}{\varepsilon^2} \right) \right) \tag{3}$$

where K_1 is a smooth and positive function; $\eta(z, .)$ is a centered stochastic process bounded by a constant strictly less than one. We will assume that $\eta(z, .)$ is stationary and has good mixing properties. Furthermore, we will assume that the correlation coefficient $\alpha(z) = \int_0^{+\infty} E\{\eta(z, 0)\eta(z, s)\}\, ds$ of $\eta(z, .)$ is finite and strictly positive.

The function F defined in the system (1) depends on the parameter ε as follows:

$$F(t, \mathbf{x}) = \frac{1}{\varepsilon^3} f\left(\frac{t}{\varepsilon}, \frac{\mathbf{x}}{\varepsilon} \right) \tag{4}$$

This means that we send a pulse of duration a wavelength. The term $\frac{1}{\varepsilon^3}$ is just a normalization factor for the reflected velocity.

2.3. Macroscopic Quantities

Let $h = h(t, \mathbf{x}, z)$ be an integrable function. We define a specific Fourier transform in time and transverse space variables for our problem by:

$$\hat{h}(\omega, \kappa, z) = \int e^{i\omega(t-\kappa.\mathbf{x})} h(t, \mathbf{x}, z)\, dt\, d\mathbf{x} \tag{5}$$

where $\kappa = (\kappa_1, \kappa_2)$ represents the horizontal slowness vector.

This Fourier transform allows the separation of plane waves.

In order to simplify the equations of this problem, it is convenient to introduce the acoustic speed c, the impedance ζ and the travel time τ:

$$c(z) = \begin{cases} \sqrt{\dfrac{K_0}{\rho_0}} & \text{if} \quad z > 0 \\ \sqrt{\dfrac{K_1(z)}{\rho_1}} & \text{if} \quad -L < z < 0 \\ \sqrt{\dfrac{K_2}{\rho_2}} & \text{if} \quad z < -L \end{cases} \tag{6}$$

$$\zeta(z, \kappa) = \frac{\rho(z)\, c(z)}{\sqrt{1 - c^2(z)\, |\kappa|^2}} \tag{7}$$

$$\tau(z, \kappa) = \int_0^z \frac{\sqrt{1 - c^2(s)\, |\kappa|^2}}{c(s)}\, ds \tag{8}$$

where $|\kappa|$ stands for the euclidian norm of the horizontal slowness vector.

As the time-reversal mirror doesn't keep in memory evanescent pulse,[(5)] we restrict ourselves to propagative waves. We will then assume $1 - c^2(z)\, |\kappa|^2$ greater than a positive constant.

3. AMPLITUDES EQUATIONS

3.1. Decomposition Into Upgoing and Downgoing Waves

Denote by (u_1, u_2, u_3) the components of the velocity vector u. The Fourier transform of the system (1) gives the following differential system

$$\begin{cases} \hat{u}_1 = \dfrac{\kappa_1}{\rho}\,\hat{p} \\ \hat{u}_2 = \dfrac{\kappa_2}{\rho}\,\hat{p} \\ -i\omega\rho\hat{u}_3 + \dfrac{\partial \hat{p}}{\partial z} = \hat{f}(\varepsilon\omega, \kappa)\,\delta_{(z-z_s)} \\ -i\omega\left(\dfrac{1}{K} - \dfrac{|\kappa|^2}{\rho}\right)\hat{p} + \dfrac{\partial \hat{u}_3}{\partial z} = 0 \end{cases} \tag{9}$$

Throughout the sequel, we will simply denote by u the vertical component of the velocity.

Combining the two last equations of the system (9), we obtain :

$$\frac{\partial^2}{\partial z^2} + \omega^2\left(\frac{\rho}{K} - |\kappa|^2\right)\hat{p} = \hat{f}(\varepsilon\omega, \kappa)\,\delta'_{(z-z_s)} \tag{10}$$

As in the homogeneous unidimensional case, we can write the Fourier transform of the pressure and the velocity in the form:

$$\hat{p} = \sqrt{\zeta}\,(Ae^{i\omega\tau} - Be^{-i\omega\tau}) \qquad \text{and} \qquad \hat{u} = \frac{1}{\sqrt{\zeta}}\,(Ae^{i\omega\tau} + Be^{-i\omega\tau}) \tag{11}$$

where $A = A(\omega, \kappa, z)$ and $B = B(\omega, \kappa, z)$ represent respectively the upgoing and downgoing waves amplitudes.

We will assume that waves which reach the interface $z = -L$ don't return in the random medium. More precisely, we will take

$$A(\omega, \kappa, -L) = 0$$

3.2. Equations for the Amplitudes

We will analyse our problem for high frequencies. We then introduce the functions a and b:

$$a(\omega, \kappa, z) = A\left(\frac{\omega}{\varepsilon}, \kappa, z\right) \qquad \text{and} \qquad b(\omega, \kappa, z) = B\left(\frac{\omega}{\varepsilon}, \kappa, z\right)$$

Using equations (9) and (11), we obtain the random differential equation:

$$\frac{\partial}{\partial z}\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{\varepsilon} P^{\varepsilon}(z, \omega, \kappa)\begin{pmatrix} a \\ b \end{pmatrix} + Q^{\varepsilon}(z, \omega, \kappa)\begin{pmatrix} a \\ b \end{pmatrix}$$
$$+ \frac{1}{2\sqrt{\zeta(z_s, \kappa)}} \hat{f}(\omega, \kappa)\begin{pmatrix} e^{-\frac{i\omega}{\varepsilon}\tau(z_s, \kappa)} \\ -e^{\frac{i\omega}{\varepsilon}\tau(z_s, \kappa)} \end{pmatrix}\delta_{(z-z_s)} \tag{12}$$

where P^{ε} and Q^{ε} are the 2×2 matrices:

$$P^{\varepsilon}(\omega, \kappa, z) = i\omega m^{\varepsilon}\begin{pmatrix} 1 & -e^{-\frac{2i\omega\tau}{\varepsilon}} \\ e^{\frac{2i\omega\tau}{\varepsilon}} & -1 \end{pmatrix} \quad \text{and}$$
$$Q^{\varepsilon}(\omega, \kappa, z) = \frac{1}{2\zeta}\frac{\partial\zeta}{\partial z}\begin{pmatrix} 0 & e^{-\frac{2i\omega\tau}{\varepsilon}} \\ e^{\frac{2i\omega\tau}{\varepsilon}} & 0 \end{pmatrix}$$

with

$$m^{\varepsilon} = m^{\varepsilon}(\omega, \kappa, z) = \frac{\zeta(z, \kappa)\,\eta\left(z, \frac{z}{\varepsilon}\right)}{2c^2(z)\,\rho(z)}$$

We will now derive the values of the amplitudes a and b at the interface $z = 0$.

Denote by Γ the reflection coefficient defined by

$$\Gamma(\omega, \kappa, z) = \frac{a(\omega, \kappa, z)}{b(\omega, \kappa, z)} \tag{13}$$

From equation (12), it follows that Γ satisfies the Riccati random equation:

$$\frac{\partial\Gamma}{\partial z} = -\frac{i\omega}{\varepsilon} m^{\varepsilon}\,[e^{-\frac{2i\omega\tau}{\varepsilon}} - 2\Gamma + \Gamma^2 e^{\frac{2i\omega\tau}{\varepsilon}}] + \frac{1}{2\zeta}\frac{\partial\zeta}{\partial z}\,[e^{-\frac{2i\omega\tau}{\varepsilon}} - \Gamma^2 e^{\frac{2i\omega\tau}{\varepsilon}}]$$
$$\Gamma_{|z=-L^+} = \frac{\zeta(-L^+, \kappa) - \zeta(-L^-, \kappa)}{\zeta(-L^+, \kappa) + \zeta(-L^-, \kappa)}\, e^{-\frac{2i\omega\tau(-L, \kappa)}{\varepsilon}} \tag{14}$$

The value of Γ at the interface $z = -L$ is derived from the continuity of $\hat{p}$ and $\hat{u}$ at $z = -L$ and from the condition $a(\omega, \kappa, -L) = 0$.

Using the geophysical hypothesis that $\frac{\rho_0}{\rho_1}$ is negligible, we obtain that the pressure vanishes in the homogeneous half space $\{z > 0\}$. It then follows:

$$a(\omega, \kappa, 0^+) = b(\omega, \kappa, 0^+)$$

From equation (12), we have the jump equations:

$$a(\omega, \kappa, z_s^-) = a(\omega, \kappa, z_s^+) - \frac{1}{2\sqrt{\zeta(z_s, \kappa)}} \hat{f}(\omega, \kappa)\, e^{-\frac{i\omega\tau(z_s, \kappa)}{\varepsilon}}$$

$$b(\omega, \kappa, z_s^-) = b(\omega, \kappa, z_s^+) + \frac{1}{2\sqrt{\zeta(z_s, \kappa)}} \hat{f}(\omega, \kappa)\, e^{-\frac{i\omega\tau(z_s, \kappa)}{\varepsilon}}$$

The above equations give the value of the reflection coefficient at z_s^-:

$$\Gamma(\omega, \kappa, z_s^-) = \frac{a(\omega, \kappa, z_s^+) - \frac{1}{2\sqrt{\zeta(z_s, \kappa)}} \hat{f}(\omega, \kappa)\, e^{-\frac{i\omega\tau(z_s, \kappa)}{\varepsilon}}}{b(\omega, \kappa, z_s^+) + \frac{1}{2\sqrt{\zeta(z_s, \kappa)}} \hat{f}(\omega, \kappa)\, e^{-\frac{i\omega\tau(z_s, \kappa)}{\varepsilon}}}$$

When z_s goes to zero, we have:

$$a(\omega, \kappa, 0^+) = \frac{1}{2\sqrt{\zeta(0, \kappa)}} \frac{1 + \Gamma(\omega, \kappa, 0^-)}{1 - \Gamma(\omega, \kappa, 0^-)} \hat{f}(\omega, \kappa)$$

We then deduce the boundary values:

$$\begin{pmatrix} a \\ b \end{pmatrix}_{z=0^-} = \frac{\hat{f}(\omega, \kappa)}{(1 - \Gamma(\omega, \kappa, 0^-))\sqrt{\zeta(0, \kappa)}} \begin{pmatrix} \Gamma(\omega, \kappa, 0^-) \\ 1 \end{pmatrix} \tag{15}$$

where $\Gamma(\omega, \kappa, 0^-)$ is obtained by solving equation (14).

4. TIME-REVERSAL METHOD

4.1. The New Pulse

We suppose that there is a time-reversal mirror located at the point $(\mathbf{x}, 0)$. Opening a window in time and space of size ε, we have by inverse Fourier transform, the scaled integral representation of the reflected velocity at the emerging time t

$$u(t + \varepsilon\sigma, \mathbf{x} + \varepsilon x, 0) = \frac{1}{(2\pi\varepsilon)^3} \int \frac{1}{\zeta(0, \kappa)}\, e^{-i\omega(\sigma - \kappa.\tilde{\mathbf{x}})} e^{-\frac{i\omega}{\varepsilon}(t - \kappa.\mathbf{x})} \hat{u}\left(\frac{\omega}{\varepsilon}, \kappa, 0\right) \omega^2 d\omega\, d\kappa \tag{16}$$

From equations (11) and (15), we obtain the Fourier transform of the velocity at the interface $z = 0$:

$$\hat{u}\left(\frac{\omega}{\varepsilon}, \kappa, 0\right) = \frac{1}{\sqrt{\zeta(0,\kappa)}} \frac{1+\Gamma(\omega,\kappa,0^-)}{1-\Gamma(\omega,\kappa,0^-)} \hat{f}(\omega,\kappa) \tag{17}$$

Reporting the above quantity in equation (16), the integral expression of the reflected velocity becomes:

$$\begin{aligned} u(t+\varepsilon\sigma, \mathbf{x}+\varepsilon\tilde{\mathbf{x}}, 0) = \frac{1}{(2\pi\varepsilon)^3} \int \frac{1}{\zeta(0,\kappa)} e^{-i\omega(\sigma-\kappa.\tilde{\mathbf{x}})} e^{-\frac{i\omega}{\varepsilon}(t-\kappa.\mathbf{x})} \hat{f}(\omega,\kappa) \\ \times \frac{1+\Gamma(\omega,\kappa,0^-)}{1-\Gamma(\omega,\kappa,0^-)} \omega^2 d\omega\, d\kappa \end{aligned} \tag{18}$$

We apply the time-reversal method introduced by J. F. Clouet and J. P. Fouque in ref. 4. This method consists here to take the velocity (18) or a part of it with a time direction reversed as the pulse:

$$g(\sigma, \tilde{\mathbf{x}}) = u(t-\varepsilon\sigma, \mathbf{x}+\varepsilon\tilde{\mathbf{x}}, 0)\, G(-\varepsilon^{\alpha_1}\sigma, \varepsilon^{\beta_1}\tilde{\mathbf{x}}) \tag{19}$$

where α_1 and β_1 are constants and G is a regular function bounded by one. The function G represents the part of the velocity used as the new pulse. We can take for example

$$G(t,\mathbf{x}) = \frac{1}{(2\pi r)^{\frac{3}{2}}} exp\left\{-\frac{t^2+x^2+y^2}{2r^2}\right\}$$ where r is a nonnegative real.

The propagation model for the new pulse is described by the Euler equations

$$\begin{cases} \rho(z)\dfrac{\partial \vec{u}}{\partial t} + \nabla p = \varepsilon^{\gamma_1} g\left(\dfrac{t}{\varepsilon}, \dfrac{\mathbf{x}}{\varepsilon}\right) \delta_{(z-z_s)} \vec{e} \\ \dfrac{1}{K}\dfrac{\partial p}{\partial t}(z) + \nabla \cdot \vec{u} = 0 \end{cases} \tag{20}$$

where the exponent γ_1 is a constant depending on α_1 and β_1.

The propagation source corresponds to the position of the time-reversal mirror.

4.2. The New Reflected Velocity

We deduce the integral representation of the new reflected velocity at the point $(\mathbf{x}_0, 0)$ and at the emerging time t_0 by substituting $\hat{f}$ by $\varepsilon^{\gamma_1+3}\hat{g}$ in equation (18).

$$u(t_0+\varepsilon\sigma, \mathbf{x}_0+\varepsilon\mathbf{x}, 0) = \frac{\varepsilon^{\gamma_1}}{(2\pi)^3}\int \frac{1}{\zeta(0,\kappa)} e^{-i\omega(\sigma-\kappa.\tilde{\mathbf{x}})} e^{-\frac{i\omega}{\varepsilon}(t_0-\kappa.\mathbf{x}_0)}\hat{g}(\omega,\kappa) \frac{1+\Gamma(\omega,\kappa,0^-)}{1-\Gamma(\omega,\kappa,0^-)}\omega^2 d\omega\, d\kappa \tag{21}$$

The Fourier transforms of the first reflected velocity $u(t-\varepsilon\sigma, \mathbf{x}+\tilde{\mathbf{x}}, 0)$ and of the function $G(-\varepsilon^{\alpha_1}\sigma, \varepsilon^{\beta_1}\tilde{\mathbf{x}})$ are respectively:

$$\frac{1}{\varepsilon^3} e^{\frac{i\omega}{\varepsilon}(t+\kappa.\mathbf{x})}\bar{\hat{f}}(\omega, -\kappa)\frac{1+\bar{\Gamma}(\omega,\kappa,0)}{1-\bar{\Gamma}(\omega,\kappa,0)} \quad \text{and} \quad \frac{1}{\varepsilon^{\alpha_1+\beta_1}}\bar{\hat{G}}\left(\frac{\omega}{\varepsilon}, -\frac{\kappa}{\varepsilon^{\beta_1-\alpha_1}}\right)$$

($\bar{f}$ denotes the complex conjugate of f).

The convolution of these two Fourier transforms gives:

$$\hat{g}(\omega,\kappa) = \varepsilon^{-3-\alpha_1-2\beta_1}\int \zeta^{-1}(0,\kappa_1)\, e^{\frac{i\omega_1}{\varepsilon}(t+\kappa_1.\mathbf{x})}\bar{\hat{f}}(\omega_1, -\kappa_1)\,\bar{\hat{G}}\left(\frac{\omega-\omega_1}{\varepsilon^{\alpha_1}}, -\frac{\kappa-\kappa_1}{\varepsilon^{\beta_1-\alpha_1}}\right) \times \frac{1+\bar{\Gamma}(\omega_1,-\kappa_1,0^-)}{1-\bar{\Gamma}(\omega_1,-\kappa_1,0^-)} d\omega_1\, d\kappa_1 \tag{22}$$

From equation (14), it follows that the function Γ is even in κ.

The new reflected velocity then becomes:

$$u(t_0+\varepsilon\sigma, \mathbf{x}_0+\varepsilon\tilde{\mathbf{x}}, 0) = \frac{\varepsilon^{\gamma_1-3-\alpha_1-2\beta_1}}{(2\pi)^3}\int \frac{1}{\zeta(0,\kappa_1)\,\zeta(0,\kappa_2)} e^{-i\omega_1(\sigma-\kappa_1.\tilde{\mathbf{x}})} e^{-\frac{i\omega_1}{\varepsilon}(t_0-\kappa_1.\mathbf{x}_0)} \times e^{\frac{i\omega_2}{\varepsilon}(t-\kappa_2.\mathbf{x})}\hat{f}(\omega_2, -\kappa_2)\,\bar{\hat{G}}\left(\frac{\omega_1-\omega_2}{\varepsilon^{\alpha_1}}, -\frac{\kappa_1-\kappa_2}{\varepsilon^{\beta_1-\alpha_1}}\right) \times \frac{1+\Gamma(\omega_1,\kappa_1,0)}{1-\Gamma(\omega_1,\kappa_1,0)}\frac{1+\bar{\Gamma}(\omega_2,\kappa_2,0)}{1-\bar{\Gamma}(\omega_2,\kappa_2,0)}\omega_1^2 d\omega_1\, d\omega_2\, d\kappa_1\, d\kappa_2 \tag{23}$$

5. ASYMPTOTIC ANALYSIS

5.1. Refocusing

In this subsection, we give refocusing properties of acoustical pulses travelling a layered random medium. These properties are:

Theorem 5.1. The new reflected pulse is only observable at the initial source and after a time equal to the emerging time of the first reflected pulse. Furthermore, this pulse doesn't depend on:

1. the quantity of the reflected velocity used as pulse if $\alpha_1 < 1$.
2. the time necessary to go to the TRM and to reach the source if $\alpha_1 > 1$.
3. the time and the position of the TRM at the interface $z = 0$ if $\alpha_1 > 1$ and $\beta_1 - \alpha_1 > 1$.

Proof. We will assume the convergence in law of the process $\frac{1+\Gamma}{1-\Gamma}$ to a regular process (this result is established in the forthcoming section).

Let:

$$\omega_1 = \omega - \tfrac{1}{2}\,\varepsilon^{p_1}h; \qquad \omega_2 = \omega + \tfrac{1}{2}\,\varepsilon^{p_1}h; \qquad \kappa_1 = \kappa - \tfrac{1}{2}\,\varepsilon^{q_1}\lambda; \qquad \kappa_2 = \kappa + \tfrac{1}{2}\,\varepsilon^{q_1}\lambda$$

where

$$p_1 = max\{1, \alpha_1\}, \qquad \text{and} \qquad q_1 = max\{1, \beta_1\}$$

Using this change of variables, we have:

$$\begin{aligned}
u(t_0+\varepsilon\sigma, \mathbf{x}_0+\varepsilon\tilde{\mathbf{x}}, 0) &= \frac{1}{(2\pi)^3}\int \zeta^{-1}\left(0, \kappa - \frac{1}{2}\,\varepsilon^{q_1}\lambda\right) \\
&\times \zeta^{-1}\left(0, \kappa + \frac{1}{2}\,\varepsilon^{q_1}\lambda\right) e^{-\frac{i\omega}{\varepsilon}(t_0 - t - \kappa.(\mathbf{x}_0+\mathbf{x}))} \\
& e^{-i\omega(\sigma-\kappa.\tilde{\mathbf{x}}) - \frac{1}{2} i\varepsilon^{q_1-1}\omega\lambda.(\mathbf{x}_0-\mathbf{x})} e^{\frac{i\varepsilon^{p_1-1}h}{2}(t_0+t-\kappa.(\mathbf{x}_0-\mathbf{x}))} \\
& exp\left\{-\frac{1}{2}\,i\varepsilon^{q_1}\omega.\lambda.\tilde{\mathbf{x}} + \frac{1}{2}\,i\varepsilon^{p_1}h\left[\sigma - \left(\kappa - \frac{1}{2}\,\varepsilon^{q_1}\lambda\right).\tilde{\mathbf{x}} + \frac{1}{2}\,\varepsilon^{q_1-1}\lambda.(\mathbf{x}_0+\mathbf{x})\right]\right\} \\
& \hat{f}\left(\omega + \frac{1}{2}\,\varepsilon^{p_1}h, -\kappa - \frac{1}{2}\,\varepsilon^{q_1}\lambda\right) \frac{1+\Gamma(\omega - \frac{1}{2}\,\varepsilon^{p_1}h, \kappa - \frac{1}{2}\,\varepsilon^{q_1}\lambda, 0)}{1-\Gamma(\omega - \frac{1}{2}\,\varepsilon^{p_1}h, \kappa - \frac{1}{2}\,\varepsilon^{q_1}\lambda, 0)} \\
& \hat{G}(\varepsilon^{p_1-\alpha_1}h, \varepsilon^{q_1-\beta_1+\alpha_1}\lambda)\, \frac{1+\bar{\Gamma}(\omega + \frac{1}{2}\,\varepsilon^{p_1}h, \kappa + \frac{1}{2}\,\varepsilon^{q_1}\lambda, 0)}{1-\bar{\Gamma}(\omega + \frac{1}{2}\,\varepsilon^{p_1}h, \kappa + \frac{1}{2}\,\varepsilon^{q_1}\lambda, 0)} \\
&\times \left(\omega - \frac{1}{2}\,\varepsilon^{p_1}h\right)^2 d\omega\, dh\, d\kappa\, d\lambda
\end{aligned} \tag{24}$$

By the linearity of the system (20), we can take $\int G(t, \mathbf{x})\, dt\, d\mathbf{x} = 1$ without loss of generality.

When ε tends to zero, the oscillant integral (24) vanishes if $(\mathbf{x}_0, t_0)$ is different from $(-\mathbf{x}, t)$. From equation (24), the other part of the theorem is obvious.

5.2. Convergence of the Mean Velocity Field

5.2.1. The Propagator

When ε is sufficiently small, the equation (12) is reduced to:

$$\frac{\partial}{\partial z}\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{\varepsilon} P^{\varepsilon} \begin{pmatrix} a \\ b \end{pmatrix} \tag{25}$$

Let $Y^{\varepsilon} = Y^{\varepsilon}(z, \frac{\tau(z,\kappa)}{\varepsilon}, \frac{z}{\varepsilon^2}, \omega, \kappa)$ be the 2×2 matrix solution of the equation:

$$\frac{dY^{\varepsilon}}{dz} = \frac{1}{\varepsilon} P^{\varepsilon} Y^{\varepsilon} \tag{26}$$

$$Y^{\varepsilon}_{|z=0} = I \tag{27}$$

where I is the identity matrix.

The matrix Y^{ε} is the propagator or the fundamental matrix of the equation (25). This matrix has the following properties:

1.

$$Y^{\varepsilon} = \begin{pmatrix} u^{\varepsilon} & v^{\varepsilon} \\ \overline{v}^{\varepsilon} & \overline{u}^{\varepsilon} \end{pmatrix} \qquad \text{with} \qquad \|u^{\varepsilon}\|^2 - \|v^{\varepsilon}\|^2 = 1$$

2. if (a, b) satisfies (25) then:

$$Y^{\varepsilon}\begin{pmatrix} a(0) \\ b(0) \end{pmatrix} = \begin{pmatrix} a(z) \\ b(z) \end{pmatrix}$$

From the above properties of the propagator, we obtain the conservation of energy relation

$$|a(z)|^2 - |b(z)|^2 = |a(0)|^2 - |b(0)|^2$$

and the reflection coefficient at the interface $z=0$

$$\Gamma(0) = -\frac{v^{\varepsilon}(-L)}{u^{\varepsilon}(-L)}$$

Consequently, the modulus of Γ is less than one.

The moment of order n of the velocity depends on $2n$ frequencies. For the analysis of this quantity, we define the $4n$-dimensional propagator $\mathscr{Y}^{\varepsilon}$:

$$\mathscr{Y}^{\varepsilon} = \begin{pmatrix} Y_1^{\varepsilon} & & & \\ & Y_2^{\varepsilon} & & \\ & & \ddots & \\ & & & Y_{2n}^{\varepsilon} \end{pmatrix}$$

with $Y_j^{\varepsilon} = Y^{\varepsilon}(z, \omega_{j'} + \frac{1}{2}(-1)^j \varepsilon h_{j'}, \kappa_{j'} + \frac{1}{2}(-1)^j \varepsilon \lambda_{j'})$ where the index j' is the entire part of $\frac{j+1}{2}$. The matrix $\mathscr{Y}^{\varepsilon}$ is the solution of the linear differential equation:

$$\frac{d\mathscr{Y}^{\varepsilon}}{dz} = \frac{1}{\varepsilon}\mathscr{P}^{\varepsilon}\mathscr{Y}^{\varepsilon}$$
$$\mathscr{Y}^{\varepsilon}|_{z=0} = I \tag{28}$$

where $\mathscr{P}^{\varepsilon}$ is the $4n \times 4n$ matrix:

$$\mathscr{P}^{\varepsilon} = \begin{pmatrix} P_1^{\varepsilon} & & & \\ & P_2^{\varepsilon} & & \\ & & \ddots & \\ & & & P_{2n}^{\varepsilon} \end{pmatrix}$$

with

$$P_j^{\varepsilon} = P^{\varepsilon}(z, \omega_{j'} + \tfrac{1}{2}(-1)^j \varepsilon h_{j'}, \kappa_{j'} + \tfrac{1}{2}(-1)^j \varepsilon \lambda_{j'})$$

We will analyse only the first two moments of the reflected velocity. We will then take $n=2$.

5.2.2. Limiting Infinitesimal Generator

Let μ_1, μ_2 and μ_3 be the Pauli matrices:

$$\mu_1 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}; \qquad \mu_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \qquad \mu_3 = \frac{1}{2}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

The matrices P_j^ε belong to the Lie algebra generated by μ_1, μ_2 and μ_3

$$P_j^\varepsilon = \sum_{k=1}^{3} M_k^j \mu_k + \varepsilon \sum_{k=1}^{3} N_k^j \mu_k \tag{29}$$

where

$$\begin{aligned}
M_1^j &= 2\omega_{j'} m^\varepsilon(z, \kappa_{j'}) \\
M_2^j &= -2\omega_{j'} m^\varepsilon(z, \kappa_{j'}) \, sin(\Phi^\varepsilon(\omega_{j'}, \kappa_{j'}, \lambda_{j'}, h_{j'})) \\
M_3^j &= -2\omega_{j'} m^\varepsilon(z, \kappa_{j'}) \, cos(\Phi^\varepsilon(\omega_{j'}, \kappa_{j'}, \lambda_{j'}, h_{j'})) \\
N_1^j &= (-1)^j \left(\omega_{j'} \frac{\nabla_\kappa \tau_{j'}}{\tau_{j'}} . \lambda_{j'} + h_{j'} \right) m^\varepsilon(z, \kappa_{j'}) \\
N_2^j &= (-1)^{j+1} \left(\omega_{j'} \frac{\nabla_\kappa \tau_{j'}}{\tau_{j'}} . \lambda_{j'} + h_{j'} \right) m^\varepsilon(z, \kappa_{j'}) \, sin(\Phi^\varepsilon(\omega_{j'}, \kappa_{j'}, \lambda_{j'}, h_{j'})) \\
N_3^j &= (-1)^{j+1} \left(\omega_{j'} \frac{\nabla_\kappa \tau_{j'}}{\tau_{j'}} . \lambda_{j'} + h_{j'} \right) m^\varepsilon(z, \kappa_{j'}) \, cos(\Phi^\varepsilon(\omega_{j'}, \kappa_{j'}, \lambda_{j'}, h_{j'}))
\end{aligned}$$

with

$$\Phi^\varepsilon(\omega_{j'}, \kappa_{j'}, \lambda_{j'}, h_{j'}) = \frac{2\omega_{j'}\tau_{j'}}{\varepsilon} + (-1)^j \omega_{j'} \lambda_{j'} . \nabla_\kappa \tau_{j'} + (-1)^j \, h_{j'} \tau_{j'}$$

$$\tau_{j'} = \tau(z, \kappa_{j'}) \qquad \text{and} \qquad \nabla_\kappa = \left(\frac{\partial}{\partial \kappa_1}, \frac{\partial}{\partial \kappa_2} \right).$$

We can then write equation (28) in the form:

$$\frac{d\mathcal{Y}^\varepsilon}{dz} = \frac{1}{\varepsilon} F\left(\mathcal{Y}^\varepsilon, z, \frac{\tau}{\varepsilon}, \frac{z}{\varepsilon^2} \right) + G\left(\mathcal{Y}^\varepsilon, z, \frac{\tau}{\varepsilon}, \frac{z}{\varepsilon^2} \right) \tag{30}$$

where F and G are the centered random matrices:

$$F\left(\mathcal{Y}^{\varepsilon}, z, \frac{\tau}{\varepsilon}, \frac{z}{\varepsilon^2}\right) = \begin{pmatrix} \sum_{k=1}^{3} M_k^1 \mu_k & & & \\ & \sum_{k=1}^{3} M_k^2 \mu_k & & \\ & & \sum_{k=1}^{3} M_k^3 \mu_k & \\ & & & \sum_{k=1}^{3} M_k^4 \mu_k \end{pmatrix} \mathcal{Y}^{\varepsilon}$$

$$G\left(\mathcal{Y}^{\varepsilon}, z, \frac{\tau}{\varepsilon}, \frac{z}{\varepsilon^2}\right) = \begin{pmatrix} \sum_{k=1}^{3} N_k^1 \mu_k & & & \\ & \sum_{k=1}^{3} N_k^2 \mu_k & & \\ & & \sum_{k=1}^{3} M_k^3 \mu_k & \\ & & & \sum_{k=1}^{3} M_k^4 \mu_k \end{pmatrix} \mathcal{Y}^{\varepsilon}$$

Let u^{ε} be defined by:

$$u^{\varepsilon}(z, Y_1, Y_2, Y_3, Y_4) = E\{g(Y_1^{\varepsilon} Y_1, Y_2^{\varepsilon} Y_2, Y_3^{\varepsilon} Y_3, Y_4^{\varepsilon} Y_4)\}$$

By the application of the approximation-diffusion theorem given in the appendix, u^{ε} converges, when ε goes to zero, to the function u solution of the backward equation:

$$\frac{\partial u}{\partial z} + \mathcal{L}_z u = 0$$

$$u(0, Y_1, Y_2, Y_3, Y_4) = g(Y_1, Y_2, Y_3, Y_4)$$

where $\mathcal{L}_z = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3$ is the infinitesimal generator:

$$\mathcal{L}_1 = \frac{\alpha(z)}{2}\left(\frac{\omega_1 \zeta(z, \kappa_1)}{c^2(z)\rho(z)}\right)^2 [2D_1^{(1)} D_1^{(1)} + 4D_1^{(1)} D_1^{(2)} + 2D_1^{(2)} D_1^{(2)} + D_2^{(1)} D_2^{(1)} + D_2^{(2)} D_2^{(2)}$$
$$+ D_3^{(1)} D_3^{(1)} + 2\cos(2(\omega_1 \lambda_1 . \nabla_{\kappa} \tau(z, \kappa_1) + h_1 \tau(z, \kappa_1)))(D_2^{(1)} D_2^{(2)} + D_3^{(1)} D_3^{(2)})]$$

$$\mathscr{L}_2 = \frac{\alpha(z)}{2}\left(\frac{\omega_2\zeta(z,\kappa_2)}{c^2(z)\,\rho(z)}\right)^2 [2D_1^{(3)}D_1^{(3)}+4D_1^{(3)}D_1^{(4)}+2D_1^{(4)}D_1^{(4)}+D_2^{(3)}D_2^{(3)}+D_2^{(4)}D_2^{(4)}$$
$$+D_3^{(3)}D_3^{(3)}+2\cos\left(2(\omega_2\lambda_2.\nabla_\kappa\tau(z,\kappa_2)+h_2\tau(z,\kappa_2))\right)(D_2^{(3)}D_2^{(4)}+D_3^{(3)}D_3^{(4)})]$$

$$\mathscr{L}_3 = \frac{2\omega_1\omega_2\zeta(z,\kappa_1)\,\zeta(z,\kappa_2)\,\alpha(z)}{c^4(z)\,\rho^2(z)}[D_1^{(3)}D_1^{(1)}+D_1^{(3)}D_1^{(2)}+D_1^{(4)}D_1^{(1)}+D_1^{(4)}D_1^{(2)}]$$

where $D_k^{(j)}$ is the operator introduced in ref. 3:

$$D_k^{(j)}f(Y_1,Y_2,Y_3,Y_4) = \lim_{h\to 0}\frac{f(Y_1,\ldots,e^{\mu_k h}Y_j,\ldots,Y_4)-f(Y_1,Y_2,Y_3,Y_4)}{h}$$

with $k=1,\cdots,3$ and $j=1,\cdots,4$.

5.2.3. Asymptotic Variance

The asymptotic analysis of the variance of the velocity gives the following result:

Theorem 5.2. The new reflected velocity converges in quadratic mean, when ε tends to zero, to the deterministic function:

$$\lim_{\varepsilon\to 0} u(t+\varepsilon\sigma,-\mathbf{x},0) = \frac{1}{2\pi}\int e^{-i\omega\sigma}[\Lambda(h,\omega,\lambda) * G_1(h,\omega,\lambda)](t,\mathbf{x})\,d\omega$$

where

$$\Lambda(\omega,t,\mathbf{x}) = \frac{\omega^2}{(2\pi)^2}\int \zeta^{-2}(0^-,\kappa)\,\bar{\hat{f}}(\omega,-\kappa)$$
$$\left[1+4\sum_{N=1}^{+\infty}W^N(0,t+\kappa.\mathbf{x},\mathbf{x},\kappa,\omega)\right]d\kappa$$

and

$$G_1(t,\omega,\mathbf{x}) = \frac{\omega^2}{(2\pi)^2}\int e^{iht+i\omega\lambda.\mathbf{x}}\hat{G}(h,\lambda)\,dh\,d\lambda$$

with W^N solution of:

$$\frac{\partial W^N}{\partial z}+2N\tau_z\left(\frac{\partial W^N}{\partial t}+\frac{\nabla_\kappa\tau_z}{\tau_z}.\nabla_\mathbf{x}W^N\right) = \frac{\omega^2\zeta^2(z,\kappa)\,N^2\alpha(z)}{2c^4(z)\,\rho^2(z)}$$
$$\times[W^{N-1}-2W^N+W^{N+1}]$$
$$W^N|_{z=-L} = (\Gamma_I(-L))^{2N}\,\delta_t\delta_\mathbf{x}$$

This theorem means that the time-reversal method preserves the Anstey–O'doherty theory.[8]

Proof. As the modulus of the reflection coefficient is less than one, we can write the velocity as the series:

$$\lim_{\varepsilon\to 0} E\{u(t+\varepsilon\sigma, -\mathbf{x}, 0)\} = \frac{1}{(2\pi)^3}\int e^{-i\omega(\sigma-\lambda.\mathbf{x})} e^{ih(t+\kappa.\mathbf{x})}\,\zeta^{-2}(0,\kappa)$$
$$\times \bar{\hat{f}}(\omega, -\kappa)\,\hat{G}(h,\lambda)\,\omega^2$$
$$\times \lim_{\varepsilon\to 0} E\left\{1+2\sum_{N=1}^{+\infty}\left(\Gamma^N\left(\omega-\frac{\varepsilon}{2}h, \kappa-\frac{\varepsilon}{2}\lambda, 0^-\right)\right.\right.$$
$$\left.+\bar{\Gamma}^N\left(\omega+\frac{\varepsilon}{2}h, \kappa+\frac{\varepsilon}{2}\lambda, 0^-\right)\right)$$
$$+4\sum_{N,M=1}^{+\infty}\Gamma^N\left(\omega-\frac{\varepsilon}{2}h, \kappa-\frac{\varepsilon}{2}\lambda, 0^-\right)$$
$$\left.\times\bar{\Gamma}^M\left(\omega+\frac{\varepsilon}{2}h, \kappa+\frac{\varepsilon}{2}\lambda, 0^-\right)\right\} d\omega\, d\kappa\, dh\, d\lambda \tag{32}$$

Denote by $\tilde{U}^{N,M}$ the function :

$$\tilde{U}^{N,M}(z,t,\mathbf{x},\kappa,\omega) = \frac{\omega^2}{(2\pi)^2}\int e^{ih(t-(N+M)\tau)+i\omega\lambda(\mathbf{x}-(N+M)\nabla_\kappa\tau)}\hat{G}(h,\lambda)\,U^{N,M} dh\, d\lambda \tag{33}$$

with

$$U^{N,M} = \Gamma^N\left(\omega-\frac{\varepsilon}{2}h, \kappa-\frac{1}{2}\varepsilon\lambda, z\right)\bar{\Gamma}^M\left(\omega+\frac{\varepsilon}{2}h, \kappa+\frac{1}{2}\varepsilon\lambda, z\right)$$

When ε is sufficiently small, the Riccati equation (14) is reduced to:

$$\frac{\partial\Gamma}{\partial z} = -\frac{i\omega}{\varepsilon} m^{\varepsilon}[e^{-\frac{2i\omega\tau}{\varepsilon}} - 2\Gamma + \Gamma^2 e^{\frac{2i\omega\tau}{\varepsilon}}] + \text{negligible terms} \tag{34}$$

It then follows that $\tilde{U}^{N,M}$ satisfies:

$$\begin{aligned}\frac{\partial \tilde{U}^{N,M}}{\partial z} &= -\frac{i\omega}{\varepsilon} m^{\varepsilon}[N e^{\frac{-2i\omega\tau}{\varepsilon}} \tilde{U}^{N-1,M} + N e^{\frac{2i\omega\tau}{\varepsilon}} \tilde{U}^{N+1,M} \\ &\quad - M e^{\frac{2i\omega\tau}{\varepsilon}} \tilde{U}^{N,M-1} - M e^{\frac{-2i\omega\tau}{\varepsilon}} \tilde{U}^{N,M+1}] \\ &\quad \times \frac{2i\omega}{\varepsilon} m^{\varepsilon}(N-M)\, \tilde{U}^{N,M} \\ &\quad -(N+M)(\tau_z + m^{\varepsilon}) \left(\frac{\partial \tilde{U}^{N,M}}{\partial t} + \frac{\nabla_\kappa \tau_z}{\tau_z} . \nabla_{\mathbf{x}} \tilde{U}^{N,M} \right) \\ &\quad + \text{negligible terms} \\ \tilde{U}^{N,M}|_{z=-L} &= (\Gamma_I(-L))^N\, (\bar{\Gamma}_I(-L))^M\, e^{-2i\omega(N-M)\frac{\tau(-L,\kappa)}{\varepsilon}} G_1(t, \omega, \mathbf{x})\end{aligned} \tag{35}$$

where

$$G_1(t, \omega, \mathbf{x}) = \frac{\omega^2}{(2\pi)^2} \int e^{iht + i\omega\lambda . \mathbf{x}} \hat{G}(h, \lambda)\, dh d\lambda$$

This equation can be re-written as follows:

$$\begin{aligned}&\frac{\partial}{\partial z} \tilde{U}^{N,M}(z, t + (N+M)(\tau + m_1), \mathbf{x} + (N+M)(\nabla_\kappa \tau + m_2), \kappa, \omega) \\ &= -\frac{i\omega}{\varepsilon} m^{\varepsilon}[N e^{\frac{-2i\omega\tau}{\varepsilon}} \tilde{U}^{N-1,M} + N e^{\frac{2i\omega\tau}{\varepsilon}} \tilde{U}^{N+1,M} - M e^{\frac{2i\omega\tau}{\varepsilon}} \tilde{U}^{N,M-1} \\ &\quad - M e^{\frac{-2i\omega\tau}{\varepsilon}} \tilde{U}^{N,M+1}] \frac{2i\omega}{\varepsilon} m^{\varepsilon}(N-M)\, \tilde{U}^{N,M} + \text{negligible terms} \\ &\tilde{U}^{N,M}|_{z=-L} = (\Gamma_I(-L))^N\, (\bar{\Gamma}_I(-L))^M e^{-2i\omega(N-M)\frac{\tau(-L,\kappa)}{\varepsilon}}\, G_1(t, \omega, \mathbf{x})\end{aligned} \tag{36}$$

where m_1 and m_2 are the stochastic processes:

$$m_1 = m_1(\omega, \kappa, z) = \int_0^z m^{\varepsilon}(\omega, \kappa, s)\, ds$$

$$m_2 = m_2(\omega, \kappa, z) = \int_0^z m^{\varepsilon}(\omega, \kappa, s)\, \frac{\nabla_\kappa \tau_s}{\tau_s}\, ds$$

If follows that the vector X^{ε} with components $\tilde{U}^{N,M}$ satisfies an equation in the form:

$$\frac{dX^{\varepsilon}}{dz} = \frac{1}{\varepsilon} A\left(z, \frac{\tau}{\varepsilon}, \frac{z}{\varepsilon^2}\right) X^{\varepsilon} + B\left(z, \frac{\tau}{\varepsilon}, \frac{z}{\varepsilon^2}\right) X^{\varepsilon}$$

where A is a zero-mean random matrix.

Let $\tilde{W}^{N,M}$ be

$$\tilde{W}^{N,M}(z, t, \mathbf{x}, \kappa, \omega) = \lim_{\varepsilon \to 0} e^{\frac{2i\omega}{\varepsilon}(N-M)\,\tau(-L,\kappa)} E\{\tilde{U}^{N,M}(z, t, \mathbf{x}, \kappa, \omega)\}.$$

The function $\tilde{W}^{N} = \tilde{W}^{N,N}$ is the solution of the transport equation:

$$\frac{\partial \tilde{W}^N}{\partial z} + 2N\tau_z \left(\frac{\partial \tilde{W}^N}{\partial t} + \frac{\nabla_\kappa \tau_z}{\tau_z} . \nabla_{\mathbf{x}} \tilde{W}^N \right) = \frac{\omega^2 \zeta^2(z,\kappa)\, N^2 \alpha(z)}{2c^4(z)\, \rho^2(z)} \times [\tilde{W}^{N-1} - 2\tilde{W}^N + \tilde{W}^{N+1}] \quad (37)$$

$$\tilde{W}^N|_{z=-L} = (\Gamma_I(-L))^{2N}\, G_1(t, \omega, \mathbf{x})$$

Let W^N be the solution of the equation (37) with the boundary condition

$$W^N|_{z=-L} = (\Gamma_I(-L))^{2N}\, \delta_t \delta_{\mathbf{x}}$$

We have, by the linearity of (37), the reflected velocity mean:

$$\lim_{\varepsilon \to 0} E\{u(t+\varepsilon\sigma, -\mathbf{x}, 0)\} = \frac{1}{2\pi} \int e^{-i\omega\sigma} [\Lambda(h, \omega, \lambda) * G_1(h, \omega, \lambda)](t, \mathbf{x})\, d\omega \quad (38)$$

where

$$\Lambda(\omega, t, \mathbf{x}) = \frac{\omega^2}{(2\pi)^2} \int \zeta^{-2}(0^-, \kappa)\, \bar{\bar{f}}(\omega, -\kappa) \times \left[1 + 4 \sum_{N=1}^{+\infty} W^N(0, t+\kappa.\mathbf{x}, \mathbf{x}, \kappa, \omega) \right] d\kappa \quad (39)$$

From the form of the third term $\mathscr{L}_3$ of the infinitesimal generator, we establish as in ref. 3 that the variance of the reflected velocity vanishes when ε goes to zero.

6. APPENDIX

We recall a limit theorem dealing with the kind of equations satisfied by the reflection coefficient. The version we give here is the one used by Papanicolaou and his co-workers in ref. 1. A full demonstration of this kind of result is given in ref. 6.

Theorem 6.1. Let $X^\varepsilon(t)$ be a stochastic process with values in R^n satisfying the following differential equation:

$$\frac{dX^\varepsilon(t)}{dt} = \frac{1}{\varepsilon} F\left(X^\varepsilon(t), t, \frac{t}{\varepsilon}, \frac{t}{\varepsilon^2}\right) + G\left(X^\varepsilon(t), t, \frac{t}{\varepsilon}, \frac{t}{\varepsilon^2}\right)$$

$$X^\varepsilon(0) = x$$

where F and G are random functions which map $R^n \times R \times R \times R$ to R^n.

Assume:

1. $F(x, t, T, s)$ and $G(x, t, T, s)$ are stationary and mixing in s for each x, t and T fixed;
2. $F(x, t, T, s)$ and $G(x, t, T, s)$ are smooth in x;
3. $E\{F(x, t, T, s)\} = 0$

then $X^\varepsilon(t)$ converges weakly when ε tends to zero to the diffusion $X(t)$ whose infinitesimal generator is:

$$\mathcal{L}_t f(x) = \lim_{T \to +\infty} \frac{1}{T} \int_0^T \int_0^{+\infty} E\{F(x, t, \tau, 0).\nabla_x[F(x, t, \tau, s).\nabla_x f(x)]\}\, ds d\tau$$

$$+ \lim_{T \to +\infty} \frac{1}{T} \int_0^T E\{G(x, t, \tau, 0).\nabla_x f(x)\}\, d\tau$$

In particular, $u^\varepsilon(t, T, x) = E\{g(X^\varepsilon(t, T, x))\}$ converges as ε goes to 0 to $u(t, T, x)$ solution of the backward Kolmogorov equation:

$$\frac{\partial u}{\partial t} + \mathcal{L}_t u = 0 \qquad t \leqslant T$$

$$u(T, T, x) = g(x)$$

Here $X^\varepsilon(t, T, x)$ is the solution of (6.1) with $X^\varepsilon(T, T, x) = x$ and g is a smooth function.

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